

ON THE RIGIDITY OF THE COISOTROPIC MASLOV INDEX ON CERTAIN RATIONAL SYMPLECTIC MANIFOLDS

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ABSTRACT. We revisit the definition of the Maslov index of loops in coisotropic submanifolds tangent to the characteristic foliation of this submanifold. This Maslov index is given by the mean index of a certain symplectic path which is a lift of the holonomy along the loop. We prove a Maslov index rigidity result for stable coisotropic submanifolds in a broad class of ambient symplectic manifolds. Furthermore, we establish a nearby existence theorem for the same class of ambient manifolds.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. The main result of this paper is the Maslov index and symplectic area rigidity for coisotropic submanifolds in a broad class of ambient symplectic manifolds. In [Zi] and [Gi3], the Maslov index is defined for loops in coisotropic submanifolds which are tangent to the characteristic foliation of the coisotropic submanifold. The Maslov index of such a loop, $x: S^1 \rightarrow W$, is the (Conley-Zehnder) mean index Δ of a symplectic path which is a lift of the holonomy along the loop to the pull-back bundle x^*TW . Although such a lift is not unique, the coisotropic Maslov index μ is well-defined. The Maslov index is a real valued index and it generalizes the usual Lagrangian Maslov index.

With this definition of the coisotropic Maslov index, we prove a result on the Maslov class rigidity. More specifically, given a closed displaceable stable coisotropic submanifold, we show that there exists a non-trivial loop lying in the submanifold with Maslov index bounded below by 1 and above by $2n + 1 - k$, where $2n$ is the dimension of the symplectic manifold and k the codimension of the coisotropic submanifold. Moreover, the result gives bounds on the symplectic area bounded by the loop; this area is positive and bounded above by the displacement energy of the coisotropic submanifold. This result was proved by Ginzburg in [Gi3] for ambient symplectic manifolds which are symplectically aspherical. The case where the characteristic foliation is a fibration is also considered in [Zi]. In this paper, we extend the result to certain rational manifolds which need not be symplectically aspherical. In the *spherical* case, the obtained loop may be trivial with non-trivial capping. Hence, in our theorem we state conditions on the ambient manifold for which this loop is non-trivial and has the referred bounds on the Maslov index and on the symplectic area. For instance, we have non-triviality and the desired bounds when the manifold is negative monotone.

The Maslov class rigidity for Lagrangian submanifolds was originally studied by Viterbo in [Vi] for the Lagrangian torus and by Polterovich in [Po1, Po2], for instance, for monotone Lagrangian submanifolds. These results show that the Maslov class satisfies certain restrictions. Namely, the minimal Maslov number lies between 1 and $n + 1$. Audin was the first to conjecture (as far as we know) that the minimal Maslov number is 2 for the Lagrangian torus; cf. [Au]. Fukaya proved this conjecture in [Fu]. There are two methods to prove this type of results. One approach, introduced by Gromov (see [Gr]), uses holomorphic curves. This approach is the one used, for instance, by Audin and Polterovich (see also [ALP]). A different approach relies on Hamiltonian Floer homology and is found, for instance, in the work of Viterbo, Kerman and Sirikci; see also [Ke1, KS].

The proof of our result follows the method used by Ginzburg in [Gi3] which is based on the second approach mentioned above together with the stability condition and certain lower bounds on the energy estimated by Bolle in [Bo1, Bo2]. The proof also relies on a suitable action selector (introduced in [Ke1, KS]).

Furthermore, we state a theorem (and outline its prove) of dense or nearby existence, that is, a theorem which guarantees the existence of periodic orbits for a dense set of energy levels. This result is presented in [Gi2] for symplectically aspherical manifolds and as mentioned there it can be viewed as a generalization of the existence of closed characteristics on stable hypersurfaces in \mathbb{R}^{2n} , established in [HZ]. We state this nearby existence theorem for a broader class of rational symplectic manifolds.

1.2. Coisotropic Maslov Index. Let (W^{2n}, ω) be a symplectic manifold and M^{2n-k} a closed coisotropic submanifold of W of codimension k . Then $(T_p M)^\omega \subseteq T_p M$ for each $p \in M$ and, denoting by ω_M the restriction of ω to M , we note that the distribution $TM^\omega := \ker \omega_M$ on M is integrable. By the Frobenius theorem, there is a foliation \mathcal{F} (the *characteristic foliation*) on M whose tangent spaces are given by TM^ω , i.e., $T\mathcal{F} = \ker \omega_M$, and the rank of this foliation is k .

Consider a capped loop $\bar{x} = (x, u)$ tangent to $T\mathcal{F}$ and the holonomy along x

$$H_t: T^\perp \mathcal{F}_{x(0)} \rightarrow T^\perp \mathcal{F}_{x(t)}.$$

There is a symplectic vector bundle decomposition of the restriction of TW to M :

$$TW|_M = (T\mathcal{F} \oplus T^\perp M) \oplus T^\perp \mathcal{F}$$

where we identify the normal bundle $T^\perp \mathcal{F}$ to \mathcal{F} in M with $TM/T\mathcal{F}$ and the normal bundle $T^\perp M$ to M in W with TW/TM . Lift the holonomy along x to x^*TW . The capping u gives rise to a symplectic trivialization of x^*TW , unique up to homotopy, and hence this lift can be viewed as a symplectic path

$$\Psi: [0, 1] \rightarrow \mathrm{Sp}(2n).$$

Following [Zi] (see also [Gi3]) we adopt

Definition 1.1. The *coisotropic Maslov index* is defined (up to a sign) as the mean index of this path, i.e.,

$$\mu(x, u) := -\Delta(\Psi).$$

This Maslov index is real valued (see Example 1.2) and is independent of the lift of the holonomy along x . However, in general, it depends on the trivialization arising from the capping u . We refer the reader to the appendix (section 5) for the definitions of the indices. The proof that this Maslov index is well-defined can be found in [Zi]. In the appendix, for the sake of completeness, we give a direct proof of this fact.

Example 1.2. Consider the Hamiltonian defined in (\mathbb{C}^n, ω_0) by

$$H(z) := 1/2 \sum_{l=1}^n \lambda_l |z_l|^2$$

with $\lambda_l \in \mathbb{R}^+$ (where ω_0 is the standard symplectic form). The ellipsoid defined as the regular level set $H^{-1}(\{1\})$ is a hypersurface (and hence a coisotropic submanifold) of \mathbb{C}^n . For each $j = 1, \dots, n$, the loop parameterized by

$$\gamma_j(t) := (0, \dots, 0, z_j(t), 0, \dots, 0)$$

where

$$z_j(t) = e^{-i\lambda_j t} z_j$$

(with $|z_j|^2 = 2/\lambda_j$ and $t \in [0, 2\pi/\lambda_j]$) is a periodic orbit of the Hamiltonian system of H lying in $H^{-1}(\{1\})$. A calculation shows that the Maslov index of the loop (γ_j, u_j) is given by

$$\mu(\gamma_j, u_j) = -\Delta(\gamma_j, u_j) = \frac{2}{\lambda_j} \sum_{l=1}^n \lambda_l$$

where u_j is some capping of γ_j . In this case, the index is independent of the capping we use.

To compute $\mu(\gamma_j, u_j)$, we use $\Psi_t = d(\varphi_H^t)_{\gamma(0)}$ the linearized flow along γ . The foliation \mathcal{F} is formed by the integral curves of φ_H^t . See section 2.2.1 for the description of the Maslov index when the loop is a periodic orbit of a Hamiltonian.

1.3. Rigidity of the Coisotropic Maslov Index (Main Theorem). In this section, we state and discuss the main theorem of this paper.

Theorem 1.3. *Let (W^{2n}, ω) be a rational closed symplectic manifold, $M^{2n-k} \subset W^{2n}$ a closed stable displaceable coisotropic submanifold of W and \mathcal{F} its characteristic foliation.*

Assume that one of the following conditions is satisfied

- *W is negative monotone,*
- *$e(M) < \hbar$, where $e(M)$ is the displacement energy of M and \hbar is the rationality constant of W ,*
- *$2n + 1 < 2N$, where N is the minimal Chern number of W .*

Then, for all $\varepsilon > 0$, there exists a capped loop $\bar{\gamma} = (\gamma, v)$ such that γ is a non-trivial loop tangent to \mathcal{F} and

$$\begin{aligned} 1 &\leq \mu(\bar{\gamma}) \leq 2n + 1 - k, \\ 0 &< \text{Area}(\bar{\gamma}) \leq e(M) + \varepsilon, \end{aligned}$$

where $\text{Area}(\bar{\gamma}) := \int_v \omega$.

Remark 1.4. The condition that W is closed can be replaced in the theorem by geometrically bounded and wide. Recall that a symplectic manifold is said to be *wide* if it admits an arbitrarily large, compactly supported, autonomous Hamiltonian whose Hamiltonian flow has no non-trivial contractible periodic orbits of period less than or equal to one; see [Gü] for more details. The proof of the theorem in this case is essentially the same as when W is closed.

Remark 1.5. In [Gi3], Ginzburg proves Theorem 1.3 when W is symplectically aspherical; see section 2.1 for the definition.

Remark 1.6. The requirements that M is displaceable and stable are essential. For instance, a closed manifold M viewed as the zero section of its cotangent bundle T^*M is not displaceable (cf. [Gr]) and the Maslov index of a loop in M is always trivial since $\pi_2(T^*M, M) = 0$. Moreover, the assumption that M is stable cannot be entirely omitted: there exist Hamiltonian systems having no periodic orbits on a compact energy level which arise as counterexamples to the Seifert conjecture; cf. [Gi1, GG2].

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2. PRELIMINARIES

In this section we introduce the notation used throughout the paper and review some facts needed to prove the results.

2.1. Symplectic Manifolds and Hamiltonians. Let (W^{2n}, ω) be a closed rational symplectic manifold and consider an almost complex structure J on W compatible with ω , i.e., such that $\langle \xi, \eta \rangle := \omega(\xi, J\eta)$ is a Riemannian metric on W .

Recall that (W, ω) is *closed* if it is compact with no boundary and is said to be (*spherically*) *rational* if the group

$$\langle [\omega], \pi_2(W) \rangle \subset \mathbb{R}$$

formed by the integrals of ω over the spheres in W is discrete, that is,

$$\langle [\omega], \pi_2(W) \rangle = \hbar \mathbb{Z}$$

where $\hbar \geq 0$. When $\langle [\omega], \pi_2(W) \rangle = 0$ we set $\hbar = \infty$. The constant \hbar is called the *rationality constant* and it is the infimum over the symplectic areas of all nonconstant spheres in W with positive area. More explicitly,

$$\hbar := \inf_{A \in \pi_2(W)} \{ \langle \omega, A \rangle : \langle \omega, A \rangle > 0 \}.$$

Denote by $c_1 := c_1(W, J) \in H^2(W, \mathbb{Z})$ the first Chern class of W . The *minimal Chern number* of a symplectic manifold (W, ω) is the integer N which generates the discrete group $\langle c_1, \pi_2(W) \rangle \subset \mathbb{R}$ formed by the integrals of c_1 over the spheres in W , i.e.,

$$\langle c_1, \pi_2(W) \rangle = N\mathbb{Z}$$

where $N \in \mathbb{Z}^+$. When $\langle c_1, \pi_2(W) \rangle = 0$, we set $N = \infty$. The constant N is given explicitly by

$$N := \inf_{A \in \pi_2(W)} \{ \langle c_1, A \rangle : \langle c_1, A \rangle > 0 \}.$$

A symplectic manifold (W, ω) is called *monotone* (*negative monotone*) if the cohomology classes c_1 and $[\omega]$ satisfy the condition

$$c_1|_{\pi_2(W)} = \tau [\omega]|_{\pi_2(W)}$$

for some non-negative (respectively, negative) constant $\tau \in \mathbb{R}$.

The manifold (W, ω) is called *symplectically aspherical* if

$$c_1|_{\pi_2(W)} = 0 = [\omega]|_{\pi_2(W)}.$$

Notice that a symplectically aspherical manifold is monotone and a monotone (or negative monotone) manifold is rational.

All the Hamiltonians H on W considered in this paper are assumed to be compactly supported and one-periodic in time, namely,

$$H: S^1 \times W \rightarrow \mathbb{R},$$

where $S^1 = \mathbb{R}/\mathbb{Z}$, and we set $H_t = H(t, \cdot)$ for $t \in S^1$. The Hamiltonian vector field X_H of H is defined by $\iota_{X_H} \omega = -dH$. The time-one map of the flow of the Hamiltonian vector field X_H is called a *Hamiltonian diffeomorphism* and denoted by φ_H .

The composition $\varphi_H^t \circ \varphi_K^t$ of two Hamiltonian flows is again Hamiltonian and it is generated by $K \# H$ where

$$(K \# H)_t := K_t + H_t \circ (\varphi_K^t)^{-1}. \quad (2.1)$$

In general, $K \# H$ is not a one-periodic Hamiltonian. However, $K \# H$ is one-periodic if $H_0 = 0 = H_1$. This condition can be met by reparametrizing the Hamiltonian as

a function of time without changing the time-one map. Thus, in what follows, we will usually treat $K \# H$ as a one-periodic Hamiltonian.

The *Hofer norm* of a one-periodic Hamiltonian H is defined by

$$||H|| := \int_0^1 (\max_W H_t - \min_W H_t) dt.$$

The Hamiltonian diffeomorphism φ_H is said to *displace* a subset U of W if

$$\varphi_H(U) \cap U = \emptyset.$$

When such a map exists, we call U *displaceable* and define the *displacement energy* of U to be

$$e(U) := \inf\{||H|| : \varphi_H \text{ displaces } U\}$$

where $||\cdot||$ is the Hofer norm.

2.2. Capped Periodic Orbits and Floer Homology. Let $x: S^1 \rightarrow W$ be a contractible loop with capping $u: D^2 \rightarrow W$, i.e., $u|_{\partial D^2} = x$. Two cappings u and v of x are called *equivalent* if the integrals of ω and of c_1 over the sphere obtained by attaching u to v are both equal to zero. For instance, when W is symplectically aspherical, all cappings of x are equivalent. A *capped closed curve* \bar{x} is, by definition, a closed curve x equipped with an equivalence class of cappings.

2.2.1. Hamiltonian Action and the Mean Index. The *action functional* of a one-periodic Hamiltonian H on a capped closed curve $\bar{x} = (x, u)$ is defined by

$$\mathcal{A}_H(\bar{x}) := - \int_u \omega + \int_{S^1} H_t(x(t)) dt.$$

The space of capped closed curves is a covering space of the space of contractible loops and the critical points of the action functional are exactly the capped one-periodic orbits of the Hamiltonian vector field X_H . The *action spectrum* $\mathcal{S}(H)$ of H is the set of critical values of the action.

A (capped) periodic orbit \bar{x} of H is said to be *non-degenerate* if the linearized return map

$$d\varphi_H: T_{x(0)}W \rightarrow T_{x(0)}W$$

has no eigenvalues equal to one. Note that capping has no effect on degeneracy or non-degeneracy of \bar{x} .

Using a trivialization of x^*TW arising from the capping of \bar{x} , the linearized flow along x

$$d\varphi_H^t: T_{x(0)}W \rightarrow T_{x(t)}W$$

can be viewed as a symplectic path $\Phi: [0, 1] \rightarrow \text{Sp}(2n)$. The mean index of \bar{x} is defined by $\Delta(\bar{x}) := \Delta(\Phi)$; see Definition 5.3. When we need to emphasize the role of H , we write $\Delta_H(\bar{x})$. A list of properties of the mean index can be found in section 5. In general, the mean index and the action depend on the equivalence class of the capping u of the loop x . More concretely, let A be a 2-sphere and denote by $\bar{x} \# A$ the recapping of \bar{x} by attaching A . Then we have

$$\Delta(\bar{x} \# A) = \Delta(\bar{x}) - 2 \langle c_1, A \rangle \quad \text{and} \quad \mathcal{A}_H(\bar{x} \# A) = \mathcal{A}_H(\bar{x}) - \int_A \omega.$$

2.2.2. Conley-Zehnder Index and Floer Homology. Consider a non-degenerate path $\Phi: [0, 1] \rightarrow \mathrm{Sp}(2n)$, i.e., such that $\Phi(1)$ has no eigenvalues equal to one. We denote by $\mu_{\mathrm{CZ}}(\Phi)$ the *Conley-Zehnder index* of Φ . For a non-degenerate capped closed orbit $\bar{x} = (x, u)$, its Conley-Zehnder index is given by the Conley-Zehnder index of the symplectic path Φ obtained from the linearized flow $d\varphi_H^t$ and a trivialization arising from the capping u . Up to a sign, it is defined as in [Sa, SZ] and we use the normalization such that $\mu_{\mathrm{CZ}}(\bar{x}) = n$ when \bar{x} is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian; cf. [GG1].

We have the following relation between the Conley-Zehnder and mean indices for non-degenerate paths and orbits; cf. [SZ]:

$$|\Delta(\Phi) - \mu_{\mathrm{CZ}}(\Phi)| < n \quad \text{and hence} \quad |\Delta(\bar{x}) - \mu_{\mathrm{CZ}}(\bar{x})| < n. \quad (2.2)$$

Let us recall the definition of the Floer homology for a non-degenerate Hamiltonian H . The Floer chain groups are generated by the capped one-periodic orbits of H and graded by the Conley-Zehnder index. The boundary operator is defined by counting solutions of the Floer equation

$$\frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = -\nabla H_t(u)$$

with finite energy. Floer trajectories for a non-degenerate Hamiltonian H with finite energy converge to periodic orbits \bar{x} and \bar{y} as $s \rightarrow \pm\infty$ and satisfy

$$E(u) = \mathcal{A}_H(\bar{x}) - \mathcal{A}_H(\bar{y}) = \int_{-\infty}^{\infty} \int_{S^1} \left\| \frac{\partial u}{\partial s} \right\|^2 dt ds.$$

The boundary operator counts Floer trajectories converging to periodic orbits y and x as $s \rightarrow \pm\infty$ and satisfying the condition $[(\text{capping of } \bar{x}) \# u] = [\text{capping of } \bar{y}]$.

This construction extends by continuity from non-degenerate Hamiltonians to all Hamiltonians; see [Sa, SZ] for more details.

Remark 2.1. The total Floer homology is independent of the Hamiltonian and, up to a shift of the grading and the effect of recapping, is isomorphic to the homology of W . More precisely, we have

$$HF_*(H) \cong H_{*+n}(W) \otimes \Lambda$$

as graded Λ -modules; see, for instance, [GG3, MS] and references therein for details on the definition of the Novikov ring Λ . In particular, the fundamental class $[W]$ can be viewed as an element of $HF_n(H)$.

Remark 2.2. To ensure that the Floer differential is defined, throughout this paper we either assume W to be weakly monotone (see, e.g. [HS, MS, On, Sa]) or utilize the machinery of virtual cycles (see, e.g., [FO, FOOO, LT]). In our main result, one of the possible conditions on W is negative monotonicity. In this case, W^{2n} is weakly monotone if and only if $N \geq n - 2$, where N is the minimal Chern number.

2.2.3. Filtered Floer Homology and Homotopy. Let us recall the definition of the filtered Floer homology for a non-degenerate Hamiltonian H . The (total) chain Floer complex $CF_*(H) =: CF_*^{(-\infty, \infty)}(H)$ admits a filtration by \mathbb{R} . For each $b \in (-\infty, \infty]$ outside $\mathcal{S}(H)$, the chain complex $CF_*^{(-\infty, b)}(H)$ is generated by the capped

one-periodic orbits of H with action \mathcal{A}_H less than b . For $-\infty \leq a < b \leq \infty$ outside $\mathcal{S}(H)$, set

$$CF_*^{(a,b)}(H) := CF_*^{(-\infty,b)}(H)/CF_*^{(-\infty,a)}(H).$$

The boundary operator $\partial: CF_*(H) \rightarrow CF_{*-1}(H)$ descends to $CF_*^{(a,b)}(H)$ and hence the *filtered Floer homology* $HF_*^{(a,b)}(H)$ is defined.

This construction also extends by continuity to all Hamiltonians. For an arbitrary (one-periodic in time) Hamiltonian H on W , set

$$HF_*^{(a,b)}(H) := HF_*^{(a,b)}(\tilde{H}) \quad (2.3)$$

where \tilde{H} is a non-degenerate perturbation of H and $-\infty \leq a < b \leq \infty$ are outside $\mathcal{S}(H)$.

When $a < b < c$, we have $CF_*^{(b,c)}(H) = CF_*^{(a,c)}(H)/CF_*^{(a,b)}(H)$ and thus obtain the long exact sequence

$$\dots \rightarrow HF_*^{(a,b)}(H) \rightarrow HF_*^{(a,c)}(H) \rightarrow HF_*^{(b,c)}(H) \rightarrow HF_{*-1}^{(a,b)}(H) \rightarrow \dots \quad (2.4)$$

By definition, a *homotopy* of Hamiltonians on W is a family of (one-periodic in time) Hamiltonians H^s smoothly parameterized by $s \in \mathbb{R}$ and such that $H^s \equiv H^0$ when s is near $-\infty$ and $H^s \equiv H^1$ when s is near ∞ ; see [Gi2] and references therein for the definitions, properties and proofs.

Set

$$E := \int_{-\infty}^{\infty} \int_{S^1} \max_W \partial_s H_t^s dt ds.$$

For every $C \geq E$, the homotopy induces a map of the filtered Floer homology, which we denote by $\Psi_{H^0 H^1}$, shifting the action filtration by C :

$$\Psi_{H^0 H^1}: HF_*^{(a,b)}(H^0) \rightarrow HF_*^{(a+C, b+C)}(H^1). \quad (2.5)$$

Example 2.3. Let H^s be an *increasing linear homotopy* from H^0 and H^1 , i.e.,

$$H^s = (1 - f(s))H^0 + f(s)H^1$$

where $f: \mathbb{R} \rightarrow [0, 1]$ is a monotone increasing compactly supported function equal to zero near $-\infty$ and equal to one near ∞ . Since

$$E \leq \int_{S^1} \max_W (H^1 - H^0) dt, \quad (2.6)$$

we have the homomorphism $\Psi_{H^0 H^1}$ for every $C \geq \int_{S^1} \max_W (H^1 - H^0) dt$.

Furthermore, we have the following continuity property for filtered homology: let (a^s, b^s) be a family (smooth in s) of non-empty intervals such that a^s and b^s are outside $\mathcal{S}(H^s)$ for some homotopy H^s and such that (a^s, b^s) is equal to (a^0, b^0) when s is near $-\infty$ and equal to (a^1, b^1) when s is near ∞ . Then there exists an isomorphism of homology

$$HF^{(a_0, b_0)}(H^0) \xrightarrow{\cong} HF^{(a_1, b_1)}(H^1). \quad (2.7)$$

When the interval is fixed and the homotopy is monotone decreasing, the isomorphism (2.7) is in fact $\Psi_{H^0 H^1}$ which in general is not the case.

2.3. Stable Coisotropic Submanifolds and Maslov Index. In this section, we give the definition and some properties of stable coisotropic submanifolds. This class of coisotropic submanifolds was introduced in [Bo1, Bo2] and is defined as follows.

The submanifold M is said to be *stable* if there exist k one-forms $\alpha_1, \dots, \alpha_k$ on M such that

$$\text{Ker } d\alpha_i \supset \text{Ker } \omega_M \quad \text{for all } i = 1, \dots, k$$

and

$$\alpha_1 \wedge \dots \wedge \alpha_k \wedge \omega_M^{n-k} \neq 0 \quad \text{on } M.$$

Notice that this condition is rather restrictive. For instance, a stable Lagrangian submanifold is necessarily a torus and a stable coisotropic submanifold is automatically orientable. Thus, examples of stable coisotropic submanifolds include Lagrangian tori and also contact hypersurfaces. Moreover, the stability condition is closed under products. For more details, we refer the reader to [Bo1, Bo2, Gi2].

As a consequence of the Weinstein symplectic neighborhood theorem, we obtain tubular neighborhoods of stable coisotropic submanifolds:

Proposition 2.4 ([Bo1, Bo2]). *Let M^{2n-k} be a closed stable coisotropic submanifold of (W^{2n}, ω) . Then, for $r > 0$ sufficiently small, there exists a neighborhood of M in W which is symplectomorphic to*

$$U_r = \{(q, p) \in M \times \mathbb{R}^k : |p| < r\}$$

equipped with the symplectic form

$$\omega = \omega_M + \sum_{j=1}^k d(p_j \alpha_j)$$

where $p = (p_1, \dots, p_k)$ are the coordinates in \mathbb{R}^k and $|p|$ is the Euclidean norm of p .

Thus, such a neighborhood is foliated by a family of coisotropic submanifolds $M_p = M \times \{p\}$ with $p \in B_r^k := \{p \in \mathbb{R}^k : |p| < r\}$ and a leaf of the characteristic foliation on M_p projects onto a leaf of the characteristic foliation on M .

Furthermore, we have

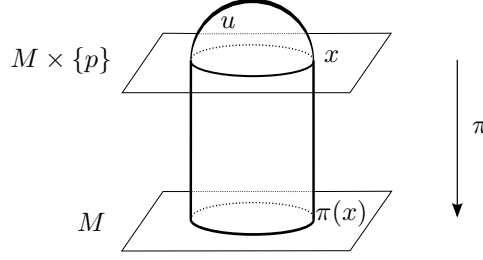
Proposition 2.5 ([Bo1, Bo2, Gi2]). *Let M^{2n-k} be a stable coisotropic submanifold of (W^{2n}, ω) . Then*

- *the leaf-wise metric $(\alpha_1)^2 + \dots + (\alpha_k)^2$ on \mathcal{F} is leaf-wise flat;*
- *the Hamiltonian flow of $\rho = (p_1^2 + \dots + p_k^2)/2$ is the leaf-wise geodesic flow of this metric.*

Consider $\bar{x} = (x, u)$ a non-trivial (capped) periodic orbit of the Hamiltonian flow of ρ . Then, as a consequence of Proposition 5.5, we obtain that the mean index $\Delta_\rho(\bar{x})$ of a periodic orbit \bar{x} of a leaf-wise geodesic flow on M is equal to, up to a sign, the coisotropic Maslov index of the projection of \bar{x} on M . More precisely,

$$\mu(\pi(x), \hat{u}) = -\Delta_\rho(x, u) \tag{2.8}$$

where \hat{u} is the capping of the orbit $\pi(x)$ given by the capping u of x together with the cylinder obtained from the projection of x on M ; see Figure 1.

FIGURE 1. Capping \hat{u} .

The following result establishes bounds on the Conley-Zehnder index of a small non-degenerate perturbation of a capped periodic orbit (x, u) of ρ which goes beyond (2.2). (Here as above M is stable.)

Proposition 2.6 ([G13]). *Let ρ' be a small perturbation of the Hamiltonian ρ defined in Proposition 2.5 and x' a non-degenerate periodic orbit of ρ' (with a capping u') close to a non-trivial periodic orbit x of ρ (with a capping u). Then*

$$\Delta_\rho(x, u) - n \leq \mu_{\text{CZ}}((x, u)') \leq \Delta_\rho(x, u) + (n - k)$$

where $(x, u)' := (x', u')$.

3. PROOF OF THE MAIN THEOREM

3.1. “Pinned” Action Selector. One of the tools used in the proof of Theorem 1.3 is an action selector defined for “pinned” Hamiltonians. This tool was first introduced in [Ke1, KS] for a class of Hamiltonians and manifolds which are somewhat different from those we work with. However, the definition of the action selector is essentially the same. In this section, we describe this action selector and a *special orbit* associated with it.

Let W be a rational symplectic manifold and U an open neighborhood of the coisotropic submanifold M of W . Consider $K: W \rightarrow \mathbb{R}$ a compactly supported autonomous Hamiltonian such that the neighborhood U contains the support of K , $\text{supp } K$, and U is displaced by a Hamiltonian H . We may assume H is non-negative with minimum value equal to zero. Suppose that K is constant on M where it attains its maximum value $\max K =: \lambda$, the maximum value λ is greater than $\|H\|$ and that K is strictly decreasing and C^2 -close to λ on a small neighborhood of M .

Consider the quotient map $j_K: HF_n(K) \rightarrow HF_n^{(\lambda-\delta, \lambda+\delta)}(K)$ and define the element $[\max_K] \in HF_n^{(\lambda-\delta, \lambda+\delta)}(K)$ as

$$[\max_K] := j_K([W])$$

where the fundamental class $[W]$ is seen as an element of $HF_n(K)$; recall Remark 2.1.

Definition 3.1 (“Pinned” Action Selector). For $\delta > 0$ small and $\alpha > \lambda + \delta$, consider the inclusion map

$$\iota_\alpha: HF_n^{(\lambda-\delta, \lambda+\delta)}(K) \hookrightarrow HF_n^{(\lambda-\delta, \alpha)}(K).$$

Define

$$c(K) := \inf_{\delta > 0} \inf \{ \alpha > \lambda + \delta : \iota_\alpha([\max_K]) = 0 \}.$$

We have $c(K) \in \mathcal{S}(K)$ and $c(K) = \mathcal{A}_K(\bar{x})$ for some capped orbit \bar{x} which is called a *special one-periodic orbit*.

Claim 3.2. *There exists $\mathcal{N} \in HF_{n+1}^{(\lambda+\delta, \infty)}(K)$ such that $\partial \mathcal{N} = [\max_K]$ where*

$$\partial: HF_{n+1}^{(\lambda+\delta, \lambda+\delta+||H||)}(K) \rightarrow HF_n^{(\lambda-\delta, \lambda+\delta)}(K)$$

is the connecting differential in the long exact sequence (2.4) (with $a = \lambda - \delta$, $b = \lambda + \delta$ and $c = \lambda + \delta + ||H||$).

Proof. For $\delta > 0$ sufficiently small, namely such that $\lambda - \delta > ||H||$, consider the following commutative diagram:

$$\begin{array}{ccc} & HF_{n+1}^{(\lambda+\delta, \lambda+\delta+||H||)}(K) & \\ & \downarrow \partial & \\ & HF_n^{(\lambda-\delta, \lambda+\delta)}(K) & \\ & \downarrow \iota & \\ HF_n^{(\lambda-\delta-||H||, \lambda+\delta)}(K) & \xrightarrow{\Psi \circ \Phi} HF_n^{(\lambda-\delta, \lambda+\delta+||H||)}(K) & \\ \Phi \downarrow & \nearrow \Psi & \\ HF_n^{(\lambda-\delta, \lambda+\delta+||H||)}(K \# H) & \xrightarrow{\Theta} HF_n^{(\lambda-\delta, \lambda+\delta+||H||)}(H) & \end{array}$$

where ι is the inclusion and ∂ is the *connecting differential* in the long exact sequence (2.4) (with $a = \lambda - \delta$, $b = \lambda + \delta$ and $c = \lambda + \delta + ||H||$). The maps Φ and Ψ are induced by monotone homotopies between K and $K \# H$: the map Φ is induced by the linear monotone increasing homotopy from K to $K \# H$ (recall that $H \geq 0$) where, in Example 2.3, $C = ||H||$; the map Ψ is induced by the linear monotone decreasing homotopy from $K \# H$ to K where, in (2.5), $C = 0$.

Since φ_H displaces $\text{supp } K$, the one-periodic orbits of $K \# H$ are exactly the one-periodic orbits of H and moreover $\mathcal{S}(K \# H) = \mathcal{S}(H)$; see [HZ]. Then the map Θ is an isomorphism induced by a linear monotone homotopy between $K \# H$ and H due to the continuity property (2.7) of filtered homology.

Note that the vertical part of the diagram, which consists of the maps ∂ and ι , is part of a long exact sequence as in (2.4).

Consider the projection

$$j_H: HF(H) \rightarrow HF^{(\lambda-\delta, \lambda+\delta+||H||)}(H).$$

and the image

$$j_H([W]) \in HF^{(\lambda-\delta, \lambda+\delta+||H||)}(H)$$

of the class $[W] \in HF_n(H)$. Since

$$\lambda - \delta > ||H||,$$

we have

$$0 = j_H([W]) \in \mathrm{HF}_n^{(\lambda-\delta, \lambda+\delta+||H||)}(H)$$

and hence

$$\mathrm{HF}_n^{(\lambda-\delta, \lambda+\delta+||H||)}(K) \ni \Psi \circ \Theta^{-1} \circ j_H([W]) = \iota([\max_K]) = 0$$

where the first equality follows from the fact that $j_H([W])$ is equal to the image $\Theta \circ \Phi \circ j([W])$ of the class $[W]$ seen as an element of $HF_n(K)$ and the map j is the projection

$$j: HF_n(K) \rightarrow HF_n^{(\lambda-\delta-||H||, \lambda+\delta)}(K).$$

Then

$$0 = [\max_K] \in \mathrm{HF}_n^{(\lambda-\delta, \lambda+\delta)}(K)$$

and, since ι and ∂ are part of a long exact sequence, it follows that there exists $\mathcal{N} \in \mathrm{HF}_{n+1}^{(\lambda+\delta, \lambda+\delta+||H||)}(K)$ such that

$$\partial \mathcal{N} = [\max_K] \in \mathrm{HF}_n^{(\lambda-\delta, \lambda+\delta)}(K).$$

□

Consider a small non-degenerate perturbation $K': S^1 \times W \rightarrow W$ of K with $\max K' = \lambda$ and such that

$$HF_j^{(a_0, a_1)}(K) := HF_j^{(a_0, a_1)}(K') \quad (3.1)$$

with $a_0, a_1 \notin \mathcal{S}(K), \mathcal{S}(K')$; recall definition (2.3).

Consider the class $[\max_{K'}] := j_{K'}([W]) \in HF_n^{(\lambda-\delta, \lambda+\delta)}(K')$ and define

$$c(K') := \inf_{\delta > 0} \inf \{ \alpha > \lambda + \delta : \iota_\alpha([\max_{K'}]) = 0 \}.$$

where $\iota_\alpha: HF_n^{(\lambda-\delta, \lambda+\delta)}(K') \hookrightarrow HF_n^{(\lambda-\delta, \alpha)}(K')$ is the inclusion map. We have $c(K') \rightarrow c(K)$ as $K' \rightarrow K$ and $c(K') = \mathcal{A}_{K'}(\bar{x}')$ for some capped orbit \bar{x}' . A special one-periodic orbit \bar{x}' for K' is obtained explicitly the following way: by (3.1) and Claim 3.2, we obtain a class $[\bar{c}'] \in HF_{n+1}^{(\lambda+\delta, \infty)}(K')$ such that $\partial[\bar{c}'] = [\max_{K'}]$. Within each chain \bar{c}' pick a capped orbit with the largest action and then among the resulting capped orbits choose a capped orbit \bar{x}' with the least action. Moreover, we have $\mu_{\mathrm{CZ}}(\bar{x}') = n + 1$.

Remark 3.3. The orbit \bar{x}' does not have to be connected with the constant orbit (γ_p, u_p) by a Floer downward trajectory. However, there exists a capped orbit \bar{y}' with this property and such that

$$\lambda \leq \mathcal{A}_K(\bar{y}') \leq \mathcal{A}_K(\bar{x}').$$

The orbit \bar{y}' is given explicitly by the following construction: take all chains \bar{c}' such that $\partial[\bar{c}'] = [\max_{K'}]$. Within each chain consider a capped orbit connected to (γ_p, u_p) with the least action and among these orbits consider one with the least action, \bar{y}' .

For a Hamiltonian K as above, consider a sequence (K_j) such that K_j is as K' above and $K_j \rightarrow K$ as $j \rightarrow \infty$. By the Arzela-Ascoli theorem, there exists a subsequence of special one-periodic orbits \bar{x}_j which converges to an orbit \bar{x} of K which is called a *special one-periodic orbit* of K . Recall that $c(K_j) \rightarrow c(K)$ as $j \rightarrow \infty$ and $\mu_{\text{CZ}}(\bar{x}_j) = n + 1$.

The following results give upper and lower bounds for the action of a special one-periodic orbit.

Lemma 3.4. *For a special one-periodic orbit \bar{x} of K , we have the following action upper bound:*

$$\mathcal{A}_K(\bar{x}) \leq \lambda + \|H\|. \quad (3.2)$$

Proof. Since $\iota([\max_K]) = 0$ (proved in Claim 3.2), $c(K) \leq \lambda + \|H\|$. By the definition of the “pinned” action selector, we have $c(K) \geq \lambda$. Then the result follows immediately from the fact that \bar{x} is a carrier of the action selector c . \square

Lemma 3.5. *A capped loop \bar{x} as in Lemma 3.4 satisfies*

$$\mathcal{A}_K(\bar{x}) - \lambda \geq \epsilon \quad (3.3)$$

where $\epsilon > 0$ is independent of K .

Proof. Consider a sequence (K_j) as above. Let u_j be a Floer downward trajectory connecting the orbit \bar{y}_j defined in Remark 3.3 and the constant orbit (γ_p, u_p) . If $E(u_j)$ is below \hbar , then we may apply a similar argument to that in lemmas 6.2 and 6.4 in [Gi2] which draws heavily from [Bo1, Bo2] and we obtain

$$d < E(u_j) = \mathcal{A}_{K_j}(\bar{y}_j) - \mathcal{A}_{K_j}(\bar{\gamma}_p)$$

where $d > 0$ is independent of K_j . Define

$$\epsilon := \max\{\hbar, d\} > 0.$$

Then $E(u_j) = \mathcal{A}_{K_j}(\bar{y}_j) - \mathcal{A}_{K_j}(\bar{\gamma}_p) \geq \epsilon$ and, since $\mathcal{A}_{K_j}(\bar{y}_j) \leq \mathcal{A}_{K_j}(\bar{x}_j)$, it follows that

$$\mathcal{A}_{K_j}(\bar{x}_j) - \lambda \geq \epsilon. \quad (3.4)$$

Then take (3.4) to the limit when $j \rightarrow \infty$ and we obtain the desired result

$$\mathcal{A}_K(\bar{x}) - \lambda \geq \epsilon.$$

\square

3.2. Proof of Theorem 1.3. Fix R such that $U_R = M \times B_R^k$ is defined by Proposition 2.4. Consider $\epsilon > 0$ small and $0 < r < R/2$. Assume U_r is displaced by some Hamiltonian H and consider $\lambda > e(U_r)$. Let $K_{\lambda, r, \epsilon}: [0, R] \rightarrow \mathbb{R}$ be a smooth decreasing map such that

- $K_{\lambda, r, \epsilon} \geq 0$
- $K_{\lambda, r, \epsilon}(0) = \lambda$
- $K_{\lambda, r, \epsilon}$ is strictly decreasing and C^2 -close to λ on $[0, \epsilon]$
- $K_{\lambda, r, \epsilon}$ is concave on $[\epsilon, 2\epsilon]$
- $K_{\lambda, r, \epsilon}$ is linear decreasing from $\lambda - \epsilon$ to ϵ on $[2\epsilon, r - \epsilon]$

- $K_{\lambda,r,\varepsilon}$ is convex on $[r - \varepsilon, r]$
- $K_{\lambda,r,\varepsilon} \equiv 0$ on $[r, R]$.

We also denote by $K_{\lambda,r,\varepsilon}$ the Hamiltonian

$$K_{\lambda,r,\varepsilon}: W \rightarrow \mathbb{R}$$

defined by $K_{\lambda,r,\varepsilon}(|p|)$ on U_R and equal to zero outside U_R .

Fix r and consider the family of functions $K_{\lambda,\varepsilon}$ depending smoothly on the parameters λ and ε . These Hamiltonians have the same properties as the Hamiltonian K in the previous subsection.

The key to the proof, as in [Gi3], is the following result which gives the location of a sequence of special one-periodic orbits \bar{x}_i .

Lemma 3.6 ([Gi3]). *There exists $\lambda > e(U_R)$ and a sequence $\varepsilon_i \rightarrow 0$ such that a special one-periodic orbit of $K_{\lambda,\varepsilon_i}$ \bar{x}_i satisfies*

$$|p(x_i)| \in [\varepsilon_i, 2\varepsilon_i]$$

where $p = (p_1, \dots, p_k)$ are the coordinates introduced in Proposition 2.4.

Remark 3.7. In [Gi3], the result of Lemma 3.6 is proved for a class of Hamiltonians which is slightly different from the one we work with. However the above lemma holds for the same reasons as the result in the referred paper.

By Proposition 2.5, if we reparametrize \bar{x}_i and reverse its orientation, then \bar{x}_i can be viewed as a periodic orbit \bar{x}_i^- of ρ . Since the slopes of the Hamiltonians $K_{\lambda,\varepsilon_i}$ are bounded from above (for instance, by $2\lambda/r$), then (by the Arzela-Ascoli theorem) we define

$$\bar{\gamma}: = \text{limit of (a subsequence of)} (\pi(x_i^-), \widehat{u}_i^-).$$

where $\mu(\pi(x_i^-), \widehat{u}_i^-) = -\Delta_\rho(x_i^-, u_i^-)$ by (2.8). Then, by (2.2),

$$-n \leq \mu_{\text{CZ}}((x_i^-, u_i^-)') - \Delta(x_i^-, u_i^-) \leq n$$

and hence

$$\begin{aligned} -n \leq \mu(\pi(x_i^-), u_i^-) + \mu_{\text{CZ}}((x_i^-, u_i^-)') &\leq n \\ \parallel \\ -\mu_{\text{CZ}}((x_i, u_i)') &= -(n+1) \end{aligned}$$

where the first equality uses the fact that x_i is in the region where $K_{\lambda,\varepsilon_i}$ is concave, i.e., where $|p(x_i)| \in [\varepsilon_i, 2\varepsilon_i]$ and we obtain the following bounds for the Maslov index of $(\pi(x_i^-), \widehat{u}_i^-)$:

$$1 \leq \mu(\pi(x_i^-), u_i^-) \leq 2n+1. \quad (3.5)$$

Considering the limit (of a subsequence) of (3.5), we have

$$1 \leq \mu(\bar{\gamma}) \leq 2n+1. \quad (3.6)$$

By Proposition 2.4, we obtain

$$\begin{aligned} A_{K_{\lambda, \varepsilon_i}}(\bar{x}_i) &= K_{\lambda, \varepsilon_i}(\bar{x}_i) - \int_{u_i} \omega \\ &= K_{\lambda, \varepsilon_i}(\bar{x}_i) - \int_{\hat{u}_i} \omega - |p(x_i)|l(\pi(x_i)) \end{aligned} \quad (3.7)$$

where \hat{u}_i is constructed as in section 2.3; see Figure 1.

Moreover, by (3.2), (3.3) and (3.7), we have

$$0 < \epsilon \leq K_{\lambda, \varepsilon_i}(x_i^-) - \int_{\hat{u}_i^-} \omega - |p(x_i^-)|l(\pi(x_i^-)) - \lambda \leq e(U_r). \quad (3.8)$$

Since $|p(x_i^-)| \in [\varepsilon_i, 2\varepsilon_i]$, $K_{\lambda, \varepsilon_i}(x_i^-) \in [\varepsilon_i, \lambda - \varepsilon_i]$ and the sequence $l(\pi(x_i^-))$ is bounded (since the slope of $K_{\lambda, \varepsilon_i}$ is bounded), then, taking the limit (of a subsequence) of (3.8), we obtain

$$0 < \epsilon \leq \text{Area}(\bar{\gamma}) \leq e(U_r). \quad (3.9)$$

Recall that ϵ is independent of ε_i . Then, taking $r > 0$ sufficiently small, we have

$$0 < \text{Area}(\bar{\gamma}) \leq e(M) + \varepsilon.$$

Hence, we have the desired bounds for the area of $\bar{\gamma}$. To obtain the Maslov index bounds as presented in the theorem (which go beyond (3.6)), we will first prove that the orbit γ is non-trivial. Assume the contrary, that is, that γ is a trivial orbit. Then, by (3.9), the capping v of γ must be non-trivial. Recall that we have one of the following conditions:

- W is negative monotone,
- $e(M) < \hbar$,
- $2n + 1 < 2N$.

Suppose that W is negative monotone. Then, $\langle c_1, v \rangle$ and $\text{Area}(\bar{\gamma})$ have opposite signs. However, by (3.6) and (3.9), they are both positive and we obtain a contradiction. If $e(M) < \hbar$ or $2n + 1 < 2N$, we obtain contradictions by the definition of the rationality constant \hbar and (3.9) or by the definition of the minimal Chern number N and (3.6), respectively. Therefore, γ is a non-trivial orbit. Furthermore, there exists a (sub)sequence of non-trivial orbits x_i as in Lemma 3.6 which converges to γ . Then, by Proposition 2.6, we have

$$\begin{aligned} -\mu(\pi(x_i^-), u_i^-) - n &\leq \mu_{\text{CZ}}((x_i^-, u_i^-)') \leq -\mu(\pi(x_i^-), \hat{u}_i^-) + n - k \\ &\parallel \\ -\mu_{\text{CZ}}((x_i, u_i)') &= -(n + 1) \end{aligned}$$

where the first equality uses the fact that x_i is in the region where $K_{\lambda, \varepsilon_i}$ is concave, i.e., where $|p(x_i)| \in [\varepsilon_i, 2\varepsilon_i]$. Then

$$1 \leq \mu(\pi(x_i^-), \hat{u}_i^-) \leq 2n + 1 - k$$

and considering the limit (of a subsequence) we obtain the desired bounds for the Maslov index of $\bar{\gamma}$:

$$1 \leq \mu(\bar{\gamma}) \leq 2n + 1 - k.$$

4. NEARBY EXISTENCE THEOREM

The theorem given in this section guarantees the existence of periodic orbits for a dense set of energy levels in a certain class of rational symplectic manifolds. This result is proved for symplectically aspherical manifolds in [Gi2]. The structure of our proof is essentially the same as in the referred paper and the necessary changes are contained in the proof of Theorem 1.3.

Let W be a closed rational symplectic manifold and consider a map $\vec{F} = (F_1, \dots, F_k): W \rightarrow \mathbb{R}^k$ whose components F_j are Poisson-commuting Hamiltonians, i.e., $\{F_i, F_j\} = 0$ for $i \neq j$ and satisfy $dF_1 \wedge \dots \wedge dF_k \neq 0$ in M_0 where $M_a := \vec{F}^{-1}(\{a\})$, for $a \in \mathbb{R}^k$, and M_0 is a displaceable coisotropic submanifold of W with codimension k . Assume that one of the following conditions is satisfied

- W is negative monotone,
- $e(M_0) < \hbar$,
- $2n + 1 < 2N$.

Then we have the following nearby existence result.

Theorem 4.1. *For a dense set of regular values $a \in \mathbb{R}^k$ near the origin, the level set M_a carries a closed curve x (with capping u in W) tangent to the characteristic foliation \mathcal{F}_a on M_a .*

Proof. We prove the existence of an orbit (with the required properties) in a level M_a arbitrarily close to M_0 and the wanted result follows immediately. Consider $K := f(F_1, \dots, F_k)$ where $f: \mathbb{R}^k \rightarrow \mathbb{R}$ is a bump function supported in a small neighborhood of the origin in \mathbb{R}^k and such that the maximum value of f is large enough. Since the support of f is small, we may assume that the support of K is displaceable and all $a \in \text{supp } f$ are regular values of \vec{F} . Hence the coisotropic manifolds M_a are compact and close to M_0 when $a \in \mathbb{R}^k$ is near the origin. By lemmas 3.4 and 3.5, there exists a capped one-periodic orbit of K (in some regular level M_a) such that

$$\max K < A_K(\bar{x}) \leq \max K + \|H\| \quad (4.1)$$

where H displaces $\text{supp } K$. The capped orbit \bar{x} can be approximated by non-degenerate capped orbits with Conley-Zehnder index equal to $n + 1$ and hence, by (2.2), we obtain

$$1 \leq \Delta(\bar{x}) \leq 2n + 1.$$

Since one of the three conditions mentioned above is satisfied, the orbit x is non-trivial. Indeed, assume that x is a trivial orbit. Then (4.1) is equivalent to

$$0 < \text{Area}(\bar{x}) \leq e(M).$$

Then using the area and (mean) index bounds on \bar{x} and assuming one of the above three conditions, we obtain a contradiction (following the same reasoning as in section 3.2).

Furthermore, since the Hamiltonian K Poisson-commutes with all F_j , the orbit x is tangent to the characteristic foliation \mathcal{F}_a on M_a . \square

5. APPENDIX: THE COISOTROPIC MASLOV INDEX IS WELL DEFINED

The objective of this section is to revisit the definition of the coisotropic Maslov index and give a direct proof of the fact that it is well defined. As mentioned in the introduction, similar notions of index are originally considered in [Gi3, Zi].

First, we define the Maslov index of a loop of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$ where $\omega_0 := dx \wedge dy$ and (x, y) are the coordinates in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Then, we define the Maslov index of a capped loop lying in a coisotropic submanifold and tangent to the characteristic foliation of the coisotropic submanifold. We start by recalling the definition of the mean index given in [SZ]. For its construction, we need a collection of mappings given by the following theorem:

Theorem 5.1 ([SZ]). *There is a unique collection of continuous mappings*

$$\rho: \text{Sp}(V, \omega) \rightarrow S^1$$

(one for every symplectic vector space V) satisfying the following conditions:

- *Naturality: If $T: (V_1, \omega_1) \rightarrow (V_2, \omega_2)$ is a symplectic isomorphism (that is, $T^*\omega_2 = \omega_1$), then*

$$\rho(T\varphi T^{-1}) = \rho(\varphi)$$

for $\varphi \in \text{Sp}(V_1, \omega_1)$.

- *Product: If $(V, \omega) = (V_1 \times V_2, \omega_1 \times \omega_2)$, then*

$$\rho(\varphi) = \rho(\varphi_1)\rho(\varphi_2)$$

for $\varphi \in \text{Sp}(V, \omega)$ of the form $\varphi(z_1, z_2) = (\varphi_1 z_1, \varphi_2 z_2)$ where $\varphi_i \in \text{Sp}(V_i, \omega_i)$.

- *Determinant: If $\varphi \in \text{Sp}(2n) \cap O(2n) \simeq U(n)$ is of the form*

$$\varphi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix},$$

then

$$\rho(\varphi) = \det(X + iY)$$

- *Normalization: If φ has no eigenvalues on the unit circle, then*

$$\rho(\varphi) = \pm 1$$

Remark 5.2. The map $\rho: \text{Sp}(2n) \rightarrow S^1$ is given explicitly by

$$\rho(\varphi) := (-1)^{m_0} \prod_{\lambda \in \sigma(\varphi) \cap S^1 \setminus \{-1, 1\}} \lambda^{m_+(\lambda)}$$

where $\sigma(\varphi)$ is the set of eigenvalues of φ , m_0 is given by

$$m_0 := \#\{\{\lambda, \lambda^{-1}\}: \lambda \in \sigma(\varphi) \cap \mathbb{R}^-\}$$

and $m_+(\lambda)$ is some multiplicity assigned to an eigenvalue $\lambda \in S^1 \setminus \{-1, 1\}$; see page 1316 in [SZ] for the details of the definition of m_+ .

Notice that only the eigenvalues of φ on the unit circle and on the negative real axis contribute to $\rho(\varphi)$.

Then, the definition of the mean index of a path $\Psi: [0, 1] \rightarrow \text{Sp}(2n)$ is given by:

Definition 5.3 (*Mean Index*; [SZ]). Let $\Psi: [0, 1] \rightarrow \text{Sp}(2n)$ be a path of symplectic matrices. Then choose a function $\alpha: [0, 1] \rightarrow \mathbb{R}$ such that $\rho(\Psi_t) = e^{\pi i \alpha(t)}$. The *Mean index* of the path Ψ is defined by

$$\Delta(\Psi) := \alpha(1) - \alpha(0)$$

The mean index Δ has the following properties:

- (1) Homotopy Invariance: $\Delta(\Psi)$ is an invariant of homotopy of Ψ with fixed end points
- (2) Concatenation: Δ is additive with respect to concatenation of paths:

$$\Delta(\Psi) = \Delta(\Psi|_{[0,a]}) + \Delta(\Psi|_{[a,1]})$$

where $0 < a < 1$

- (3) Loop: $\Delta(\varphi\Psi) = \Delta(\varphi) + \Delta(\varphi_0\Psi)$ if either φ or Ψ is a loop
- (4) Naturality: $\Delta(T\Psi T^{-1}) = \Delta(\Psi)$ where $T: (V_1, \omega_1) \rightarrow (V_2, \omega_2)$ is a symplectic isomorphism and $\Psi \in \text{Sp}(V_1, \omega_1)$
- (5) Product: $\Delta(\Psi) = \Delta(\Psi_1) + \Delta(\Psi_2)$ where $\Psi \in \text{Sp}(V = V_1 \times V_2, \omega = \omega_1 \times \omega_2)$ is given by $\Psi(z_1, z_2) = (\Psi_1 z_1, \Psi_2 z_2)$ where $\Psi_i \in \text{Sp}(V_i, \omega_i)$.

The Maslov index of a loop of coisotropic subspaces is given (up to a sign) as the mean index of a certain path of symplectic matrices.

Definition 5.4 (*Maslov Index for Coisotropic Subspaces*; cf. [Zi]). Consider

$$\mathcal{C} = (\mathcal{C}_t)_{t \in [0,1]}$$

an oriented loop of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$ and

$$H_t: \mathcal{C}_0/\mathcal{C}_0^{\omega_0} \rightarrow \mathcal{C}_t/\mathcal{C}_t^{\omega_0}$$

a path of symplectic linear maps. Recall that a loop \mathcal{C} is oriented if one can orient the space \mathcal{C}_t (continuous in t) so that \mathcal{C}_0 and \mathcal{C}_1 have the same orientation. Pick a path

$$\Psi: [0, 1] \rightarrow \text{Sp}(2n) \quad \text{satisfying} \quad \Psi_0 = Id, \quad \Psi_t(\mathcal{C}_0) = \mathcal{C}_t \quad \text{and} \quad \Psi_t \Big|_{\mathcal{C}_0/\mathcal{C}_0^{\omega_0}} = H_t \quad (5.1)$$

and define the real valued index $\mu: \mathfrak{C} \rightarrow \mathbb{R}$ by

$$\mu(\mathcal{C}, H) := -\Delta(\Psi),$$

where \mathfrak{C} is the set of loops of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$.

If the loop \mathcal{C} is not oriented, we define the Maslov index $\mu(\mathcal{C}, H)$ as half of the Maslov index of the loop obtained by traversing the initial loop twice.

Proposition 5.5. *The Maslov index given in Definition 5.4 is well defined.*

Proof. We prove this proposition in three steps by considering the following cases:

- (1) \mathcal{C} is the constant loop where $\mathcal{C}_t = L_0$ is a fixed Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$
- (2) \mathcal{C} is the constant loop where $\mathcal{C}_t = \mathcal{C}_0$ is a fixed coisotropic subspace of $(\mathbb{R}^{2n}, \omega_0)$
- (3) General case: \mathcal{C} is a loop of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$.

Step 1: Assume, without loss of generality, that \mathcal{C} is the constant *horizontal loop* $L_0 := \{(x, y) \in \mathbb{R}^{2n} : y = 0\}$. Then consider $\Psi : [0, 1] \rightarrow \text{Sp}(2n)$ as in (5.1) and notice that since $\mathcal{C}_t = L_0$ is Lagrangian, $H \equiv 0$. For $t \in [0, 1]$, we have that Ψ_t fixes the lagrangian L_0 if and only if it is of the form

$$\begin{pmatrix} A_t & B_t \\ 0 & A_t^{-T} \end{pmatrix} \text{ where } B_t^T A_t^T = A_t^{-1} B_t.$$

This path is homotopic to the concatenation of two symplectic paths of the form:

$$\Psi'_t = \begin{pmatrix} \tilde{A}_t & 0 \\ 0 & \tilde{A}_t^{-T} \end{pmatrix} \quad \text{and} \quad \Psi''_t = \begin{pmatrix} \tilde{A}_1 & \tilde{B}_t \\ 0 & \tilde{A}_1^{-T} \end{pmatrix}$$

where we essentially first travel along Ψ_t with $B_t = 0$ and then, when we reach

$$\begin{pmatrix} A_1 & B_0 = 0 \\ 0 & A_1^{-T} \end{pmatrix},$$

we build up B_t from 0 to B_1 .

Since Ψ''_t has constant eigenvalues, $\Delta(\Psi'') = 0$. Hence, by property (2), the mean index of Ψ is equal to the mean index Ψ' .

Suppose that \tilde{A}_t is diagonalizable, i.e., it can be written in the form

$$\tilde{A}_t = P_t \underbrace{\begin{pmatrix} (A_1)_t & & 0 \\ & \ddots & \\ 0 & & (A_n)_t \end{pmatrix}}_{=: D_t} (P_t)^{-1} \quad (5.2)$$

where $P_t \in \text{Sp}(2n)$ and each block $(A_j)_t$ corresponds to an eigenvalue $(\lambda_j)_t$ of \tilde{A}_t . Then, in this case,

$$\begin{pmatrix} \tilde{A}_t & 0 \\ 0 & \tilde{A}_t^{-T} \end{pmatrix} = \begin{pmatrix} P_t & 0 \\ 0 & P_t^{-T} \end{pmatrix} \underbrace{\begin{pmatrix} D_t & 0 \\ 0 & D_t^{-T} \end{pmatrix}}_{=: \Gamma_t} \begin{pmatrix} P_t & 0 \\ 0 & P_t^{-T} \end{pmatrix}^{-1}$$

and, by the naturality property of the map ρ , we have $\rho(\Psi'_t) = \rho(\Gamma_t)$ for all $t \in [0, 1]$.

Claim 5.6. *For all $t \in [0, 1]$, we have $\rho(\Gamma_t) = 1$.*

Proof. For the sake of simplicity, we will drop, for now, the subscript t in the notation. By Remark 5.2, we have

$$\begin{aligned} \rho(\Gamma) &:= (-1)^{m_0} \prod_{\lambda \in \sigma(\Gamma) \cap S^1 \setminus \{-1, 1\}} \lambda^{m_+(\lambda)} \\ &= (-1)^{m_0} \prod_{\substack{\lambda \in \sigma(\Gamma) \cap S^1 \setminus \{-1, 1\} \\ \text{Im } \lambda > 0}} \lambda^{m_+(\lambda)} \bar{\lambda}^{m_+(\bar{\lambda})} \\ &= (-1)^{m_0} \prod_{\substack{\lambda \in \sigma(\Gamma) \cap S^1 \setminus \{-1, 1\} \\ \text{Im } \lambda > 0}} \lambda^{m_+(\lambda) - m_+(\bar{\lambda})} \end{aligned} \quad (5.3)$$

where $\sigma(\Gamma)$ is the spectrum of Γ . Recall that only the eigenvalues of Γ on the unit circle and on the negative real axis contribute to $\rho(\Gamma)$. Regarding the eigenvalues on S^1 , it can be proved, directly from the definition of m_+ , that $m_+(\lambda) = m_+(\bar{\lambda})$. Hence, using the notation with the subscript t , we obtain by (5.3) that $\rho(\Gamma_t) = (-1)^{(m_0)_t}$, for each $t \in [0, 1]$, where

$$(m_0)_t := \#\{\{\lambda_t, \lambda_t^{-1}\} \in \sigma(\Gamma_t) : \lambda_t \in \mathbb{R}^-\} = \#\{\lambda_t \in \sigma(D_t) : \lambda_t \in \mathbb{R}^-\}.$$

The last equality follows from the fact that λ_t is an eigenvalue of D_t if and only if λ_t and λ_t^{-1} are eigenvalues of Γ_t . Since D_t is continuous in t and $\det(D_t) \neq 0$, the signs of $\det(D_0)$ and $\det(D_1)$ are the same. The determinant of D_t is given by

$$\det(D_t) = \prod_{\lambda_t \in \mathbb{R}^-} \lambda_t \underbrace{\prod_{\lambda_t \in \mathbb{R}^+} \lambda_t}_{>0} \underbrace{\prod_{\lambda_t \in \mathbb{C} \setminus \mathbb{R}} \lambda_t}_{>0}$$

where the products run over $\lambda_t \in \sigma(D_t)$. Then the sign of $\det(D_t)$ is determined by the number (mod 2) of the real negative eigenvalues of D_t and we have $(-1)^{(m_0)_0} = (-1)^{(m_0)_t}$ for all $t \in [0, 1]$. Since, by (5.1) $D_0 = Id$ the result follows immediately. \square

Hence, we have proved that, under the assumption (5.2), $\rho(\Psi'_t) = 1$ for a fixed $t \in [0, 1]$. Since the set of diagonalizable matrices is dense in the set of matrices, the result holds for a “general” Ψ'_t . It follows that $\Delta(\Psi') = 0$ and hence we have $\Delta(\Psi) = 0$.

Step 2: Consider $\Psi : [0, 1] \rightarrow \text{Sp}(2n)$ as in (5.1) and the symplectic decomposition of \mathbb{R}^{2n} :

$$\mathbb{R}^{2n} = (\mathbb{R}^{2n}/\mathcal{C}_0 \oplus \mathcal{C}_0^{\omega_0}) \oplus \mathcal{C}_0/\mathcal{C}_0^{\omega_0}. \quad (5.4)$$

Since $\Psi_t \in \text{Sp}(2n)$, $\Psi_t(V) = V$ and $\Psi_t(\mathcal{C}_0/\mathcal{C}_0^{\omega_0}) = \mathcal{C}_0/\mathcal{C}_0^{\omega_0}$, the path Ψ_t has the form

$$\begin{bmatrix} (\Psi_t)|_V & 0 \\ 0 & H_t \end{bmatrix}$$

with respect to decomposition (5.4), where $V := \mathbb{R}^{2n}/\mathcal{C}_0 \oplus \mathcal{C}_0^{\omega_0}$. By property (5) of the mean index,

$$\Delta(\Psi) = \Delta(\Psi|_V) + \Delta(H).$$

Since V is symplectic and $\mathcal{C}_0^{\omega_0}$ is Lagrangian in V , we have by step 1 that $\Delta(\Psi|_V) = 0$ and hence $\Delta(\Psi) = \Delta(H)$. Therefore, the mean index $\Delta(\Psi)$ only depends on the mean index of H and the result is proved for case (2).

Step 3: Let $\Psi : [0, 1] \rightarrow \text{Sp}(2n)$ be a path as in (5.1) and consider a loop $\Phi : [0, 1] \rightarrow \text{Sp}(2n)$ which depends only on \mathcal{C} and satisfies $\Phi_t(\mathcal{C}_0) = \mathcal{C}_t$. Define the path $\tilde{\Psi} : [0, 1] \rightarrow \text{Sp}(2n)$ by $\tilde{\Psi}_t := \Phi_t^{-1}\Psi_t$ which satisfies $\tilde{\Psi}_t(\mathcal{C}_0) = \mathcal{C}_t$ for all $t \in [0, 1]$. By step 2, $\Delta(\tilde{\Psi}) = \Delta(\tilde{H})$, where $\tilde{H}_t : \mathcal{C}_0/\mathcal{C}_0^{\omega_0} \rightarrow \mathcal{C}_t/\mathcal{C}_t^{\omega_0}$ is given by

$$\tilde{H}_t = \Phi_t^{-1}|_{(\mathcal{C}_t/\mathcal{C}_t^{\omega_0})} \Psi_t|_{(\mathcal{C}_0/\mathcal{C}_0^{\omega_0})} = \Phi_t^{-1}|_{(\mathcal{C}_t/\mathcal{C}_t^{\omega_0})} H_t.$$

Since Φ is a loop, then by property (3) of the mean index we have $\Delta(\tilde{\Psi}) = \Delta(\Phi^{-1}\Psi) = \Delta(\Phi^{-1}) + \Delta(\Psi)$ and $\Delta(\tilde{H}) = \Delta(\Phi^{-1}|_{(C_0/C_0^\omega)}) + \Delta(H)$. Hence

$$\Delta(\Psi) = \Delta(\tilde{H}) - \Delta(\Phi^{-1})$$

which only depends on H and on Φ . Since Φ_t only depends on \mathcal{C}_t , $\Delta(\Psi)$ only depends on H and \mathcal{C} . Therefore, the Maslov index $\mu(\mathcal{C}, H) := -\Delta(\Psi)$ depends only on the loop $\mathcal{C} = (\mathcal{C}_t)$ and the linear map H and not on the choice of the path Ψ as long as it satisfies the properties in (5.1). \square

We, now, define the Maslov index of a capped loop lying in a coisotropic submanifold and tangent to the characteristic foliation of the coisotropic submanifold.

Definition 5.7 (*Maslov Index of a Capped Loop*). Let (W, ω) be a symplectic manifold, M^{2n-k} a coisotropic submanifold of (W, ω) and \mathcal{F} its characteristic foliation. Consider $x: S^1 \rightarrow M$ a loop in M tangent to \mathcal{F} and $u: D^2 \rightarrow W$ a capping of the loop x in W . We have the symplectic vector bundle decomposition

$$TW|_M = (TW/TM \oplus T\mathcal{F}) \oplus TM/T\mathcal{F}.$$

Assume $x^*T\mathcal{F}$ is orientable and hence trivial. Denote by ξ a trivialization of $x^*T\mathcal{F}$:

$$x^*T\mathcal{F} \xrightarrow{\xi} S^1 \times T_{x(0)}\mathcal{F}.$$

Moreover, we have the following isomorphism

$$TW/TM \cong T^*\mathcal{F},$$

and hence $\xi \oplus \xi^*$ can be viewed as a family of symplectic maps

$$\Xi_t: TW/TM_{x(0)} \oplus T_{x(0)}\mathcal{F} \rightarrow TW/TM_{x(t)} \oplus T_{x(t)}\mathcal{F}.$$

Denote by $H_t: (TM/T\mathcal{F})_{x(0)} \rightarrow (TM/T\mathcal{F})_{x(t)}$ the holonomy along x . The capping u gives rise to a symplectic trivialization, unique up to homotopy, of x^*TW . Using such a trivialization, the map $\Xi_t \oplus H_t$ can be viewed as a path

$$\Psi: [0, 1] \rightarrow \mathrm{Sp}(2n)$$

which, up to some identifications, satisfies

$$\Psi_0 = Id, \quad \Psi_t(T_{x(0)}M) = T_{x(t)}M \quad \text{and} \quad \Psi_t|_{(TM/T\mathcal{F})_{x(0)}} = H_t. \quad (5.5)$$

Define the *Maslov index* of (x, u) as $\mu(x, u) := -\Delta(\Psi)$. If $x^*T\mathcal{F}$ is not orientable, we define $\mu(x, u)$ as $\mu(x^2, u^2)/2$ where (x^2, u^2) is the double cover of (x, u) .

Remark 5.8. By Proposition 5.5, $\mu(x, u)$ is independent of the trivialization ξ . However it may depend on the capping u . We give some properties of the coisotropic Maslov index:

- Homotopy Invariance: $\mu(x, u)$ is invariant under a homotopy of x in a leaf of \mathcal{F} .
- Recapping: $\mu(x, u \# A) = \mu(x, u) + 2 \langle c_1, A \rangle$ where $u \# A$ is the notation for the recapping of (x, u) by a 2-sphere A .
- Homogeneity: $\mu(x^k, u^k) = k\mu(x, u)$ where (x^k, u^k) is the k -fold cover of (x, u) .

REFERENCES

- [Au] M. Audin, Cobordism of Lagrangian immersions in the space of the cotangent bundle of a manifold, in *Funct. Anal. Appl.*, (English edition) **21** (1988), 233–226.
- [ALP] M. Audin, J. Lafontaine, L. Polterovich, *Holomorphic curves in symplectic geometry*, Eds: M. Audin, J. Lafontaine, Progress in Mathematics, **117**, Birkhäuser Verlag, Basel, 1994, pp.271–321.
- [Bo1] P. Bolle, Une condition de contact pour les sous-varités coisotropes d’une variété symplectique, in *C. R. Acad. Sci. Paris, Series I*, **322** (1996), 83–86.
- [Bo2] P. Bolle, A contact condition for p -codimensional submanifolds of a symplectic manifold ($2 \leq p \leq n$), in *Math. Z.*, **227** (1998), 211–230.
- [Fu] K. Fukaya, Application of Floer homology of Lagrangian submanifolds to symplectic topology, *Morse theoretic methods in nonlinear analysis and in symplectic topology*, 231276, NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006.
- [FO] K. Fukaya, K. Ono, Arnold conjecture and Gromov-Witten invariant, in *Topology*, **38** (1999), 933–1048.
- [FOOO] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono, *Lagrangian Intersection Floer Theory: Anomaly and Obstruction*, Parts I and II, AMS/IP Studies in Advanced Mathematics, vol. 46.1 and 46.2, American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.
- [Gi1] V.L. Ginzburg, Hamiltonian dynamical systems without periodic orbits, in *Northern California Symplectic Geometry Seminar*; Eds: Y. Eliashberg, D. Fuchs, T. Ratiu, A. Weinstein; Amer. Math. Soc. Transl. Ser. 2, **196**, AMS, Providence, RI (1999), 35–48.
- [Gi2] V.L. Ginzburg, Coisotropic intersections, *Duke Math. J.*, **140** (2007), 111–163.
- [Gi3] V.L. Ginzburg, On the Maslov class rigidity for coisotropic submanifolds, *Pacific J. Math.*, **250** (2011), No. 1, 139161.
- [Gü] B.Z. Gürel, Totally non-coisotropic displacement and its applications to Hamiltonian dynamics, *Comm. Contemp. Math.*, **10** (2008), no. 6, 1103–1128.
- [GG1] V.L. Ginzburg, B.Z. Gürel, Action and index spectra and periodic orbits in Hamiltonian dynamics, *Geom. Topol.* (2009), **13** (2009), 2745–2805.
- [GG2] V.L. Ginzburg, B.Z. Gürel, A C^2 -smooth counterexample to the Hamiltonian Seifert conjecture in \mathbb{R}^4 , *Ann. of Math.*, **158** (2003), 953–976.
- [GG3] V.L. Ginzburg, B.Z. Gürel, Conley conjecture for negative monotone symplectic manifolds, Preprint 2010, arXiv:1011.5018.
- [GK] V.L. Ginzburg, E. Kerman, Homological resonances for Hamiltonian diffeomorphisms and Reeb flows, *Int. Math. Res. Not. IMRN*, **1** (2010), 53–68.
- [Gr] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, *Invent. Math.*, **82** (1985).
- [HS] H. Hofer, D. Salamon, Floer homology and Novikov rings, in *The Floer Memorial Volume*, 483–524, Progr. Math., **133**, Birkhäuser, Basel, 1995.
- [HZ] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, 1994.
- [Ke1] E. Kerman, Action selectors and Maslov class rigidity, *Int. Math. Res. Not. IMRN* (2009); doi:10.1093/imrn/rnp093.
- [KS] E. Kerman, N.I. Siricki, Maslov class rigidity for Lagrangian submanifolds via Hofer’s geometry, *Commentarii Mathematici Helvetici* (2008) **85**, 907–949.
- [LT] G. Liu, G. Tian, Floer homology and Arnold conjecture, *J. Differential Geom.* (1998) **49**, 1–74.
- [MS] D. McDuff, D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Colloquium publications, vol. 52, AMS, Providence, RI, 2004.
- [On] K. Ono, On the Arnold conjecture for weakly monotone symplectic manifolds, *Invent. Math.* (1995) **119**, 519–537.
- [Po1] L. Polterovich, The Maslov class of the Lagrange surfaces and Gromov’s pseudoholomorphic curves, *Trans. Amer. Math. Soc.*, **325** (1991), 241–248.

- [Po2] L. Polterovich, Monotone Lagrange submanifolds of linear spaces and the Maslov class in cotangent bundles, *Math. Z.*, **207** (1991), 217–222.
- [Sa] D.A. Salamon, Lectures on Floer homology, in *Symplectic Geometry and Topology*, Eds: Y. Eliashberg and L. Traynor, IAS/Park City Mathematics series, **7** (1999), pp. 143–230.
- [SZ] D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.*, **45** (1992), 1303–1360.
- [U1] M. Usher, Boundary depth in Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds, Preprint 2009, arXiv:0903.0903; to appear in *Israel J. Math.*.
- [U2] M. Usher, Spectral numbers in Floer theories, *Compositio Mathematica*, **144** (2008), no. 6, 1581–1592.
- [Vi] C. Viterbo, A new obstruction to embedding Lagrangian tori, *Invent. Math.*, **100** (1990), 301–320.
- [Zi] F. Ziltener, A Maslov map for coisotropic submanifolds, leaf-wise fixed points and presymplectic non-embeddings, Preprint 2009, arXiv:0911.1460.

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